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An Analysis of Flat-Plate Drag with Polymer Additives

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The pipe friction reduction of aqueous polymer solutions is reviewed briefly, and the work of Meyer in correlating polymer velocity profiles is discussed. Meyer's analytical expression, which contains two polymer constants α and v_0^*, is interpreted to show that the pipe friction reduction is due to an effective increase of the Newtonian Reynolds number. Meyer's correlation is then assumed to hold for boundary-layer flow, from which an exact expression is derived for flat-plate skin friction with polymer solutions. The two significant parameters are the constant α and the speed ratio (U/v_0^*). Plotted results show the friction reduction to be confined to the plate's leading edge. The reduction is shown to be due to a Newtonian Reynolds number increase exactly equivalent to that of the pipe. Some total drag results are plotted, and an extension of the method to flows with pressure gradient is briefly outlined.

Nomenclature

A = constant defined by Eq. (21)
B = constant in the law of the wall, Eq. (1)
C_f = local plate-skin friction = $2\tau_w/\rho U^2$
C_D = plate drag coefficient = $2(\text{drag})/\rho U^2 L$
f = pipe friction factor = $2\tau_w/\rho\bar{u}^2$
k = mixing length constant $\doteq 0.4$
p = freestream static pressure
R = pipe Reynolds number = $\bar{u}D/\nu$
R_X = Ux/ν = local plate Reynolds number
R_L = UL/ν = plate length Reynolds number
U = freestream velocity
u, v = velocities parallel and normal to the plate
v^* = wall friction velocity = $(\tau_w/\rho)^{1/2}$
x, y = coordinates parallel and normal to the plate
α = polymer constant defined by Eq. (2)
β = Clauser's dimensionless pressure gradient, Eq. (23)
δ = boundary-layer thickness
ν = fluid kinematic viscosity
ξ = friction variable = $U/v^* = (2/C_f)^{1/2}$
ρ = fluid density
τ = shear stress in boundary layer

Subscripts

0 = threshold value where polymer activates
N = value for a Newtonian fluid

Superscript

$+$ = law of the wall variable, Eq. (1)

Introduction

THE well-known friction reduction property of dilute polymer additives in water has been the subject of considerable study.[1-18] For reasons of convenient experimentation, most reported data have been for pipe flow. Theoretical studies have been mostly confined to laminar flow of power-law and "second-order" fluids[2] and to turbulent flow by similarity considerations.[3,19,22]

If these additives are to achieve their predicted importance in marine applications, it will be necessary to know their effects on external body flows, i.e., boundary-layer flows.[21,23] Such flows are primarily turbulent, and the pipe flow simplifications of constrained thickness and known shear stress distribution are not applicable.

This paper will present an analysis of turbulent boundary-layer flow of a dilute polymer solution. Although its emphasis is on flat-plate flow, where the analysis is particularly simple, the extension to arbitrary pressure gradient is also outlined. Some results contradict previous estimates and place a definite limit on boundary-layer drag reduction. When compared to pipe flow, the boundary-layer calculations are in general rather unusual, in that the author had not expected them in advance.

Pipe Flow Studies

The reduction of pipe friction by dilute polymer additives is now being discussed in the popular scientific literature.[4] Early workers[5,6] thought that the friction reduction was due to strong non-Newtonian viscoelastic effects of the additive. Although heavy concentrations of additive do exhibit such effects, it is now thought that these actions are secondary. For example, Hoyt and Fabula[7] report substantial friction reductions for such minute dilution (5 ppm or less) that the fluid is essentially Newtonian.

After careful correlation of many experiments, Meyer[8] showed that drag reduction in pipes is due, instead, to thickening of the laminar sublayer. Quantitatively, the familiar law of the wall for turbulent flow,

$$u^+ = u/v^* = (1/k)\ln(yv^*/\nu) + B \tag{1}$$

where $v^* = (\tau_w/\rho)^{1/2}$ is the wall friction velocity, is altered by a dilute additive only in an increase in the constant B. The mixing-length constant k, which equals approximately 0.4, is apparently unaffected by dilute concentrations of additive. Meyer further correlated the constant B and showed it to remain equal to the Newtonian value ($B \doteq 5.5$) until the friction velocity reaches a threshold value v_0^*, after which B increases approximately logarithmically with v^*:

$$B \doteq 5.5 + \alpha\ln(v^*/v_0^*) \tag{2}$$

where α is a dimensionless constant dependent upon the type of additive and the (percentage) concentration. The threshold friction velocity v_0^*, which is of the order of a few inches per second, depends upon the type of additive but not its concentration. Meyer suggests from data in the literature that v_0^* should be approximately inversely proportional to the radius of gyration of the polymer molecule. In any case, α and v_0^* are "properties" of a given solution, and Eq. (2) may be thought of as a constitutive relation for turbulent

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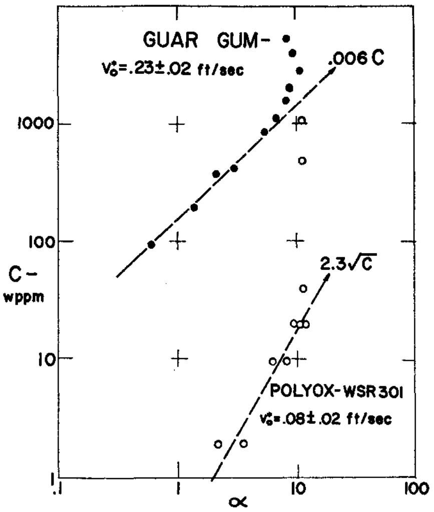

Fig. 1 Measured values of α and v_0^* for two types of polymer additives.[6,14,5,9,26]

flow, which enables the present boundary-layer analysis to be made.

Some values of α and v_0^* for various polymers are given in Fig. 1. Note that simple curve fit expressions are obtained for both of the polymer types. Based on such data that are available, it appears that the maximum value of α is approximately 11. It is also of interest that the guar gum data in Fig. 1 is for three different commercial natural gums, Jaguar, CMC-70, and J-2P, as given by Refs. 5, 6, and 9.

Using Eqs. (1) and (2) and the known linear shear distribution for fully developed pipe flow, Meyer derived the following modified friction factor relation for Reynolds numbers large enough that the wall friction exceeds v_0^*:

$$f^{-1/2} = (4 + \alpha/2^{1/2}) \log(Rf^{1/2}) - 0.394 - (\alpha/2^{1/2}) \log(2^{1/2}v_0^*D/\nu) \quad (3)$$

where f is the friction factor and R the diameter Reynolds number. When $\alpha = 0$, Eq. (3) reduces to the familiar Newtonian relation for smooth pipe flow. Equation (3) is in excellent agreement with experiment for dilute additives, as shown by Meyer[8] and later by Ernst.[9] The pipe friction follows the Newtonian relation up to a critical Reynolds number R_0, where $v^* = v_0^*$, after which the friction decreases at a sharply greater rate. Shown in Fig. 2 are the plots of $f^{-1/2}$ vs $Rf^{1/2}$ suggested by Eq. (3). The straight line for the polymer has a higher slope than the Newtonian curve. As next discussed, Eq. (3) is subject to two interesting interpretations which are useful in the flat-plate analysis to be presented.

First, if we regroup the terms, Eq. (3) becomes

$$f^{-1/2} = f_N^{-1/2} + \Delta B/2^{1/2} \quad (4)$$

where f_N denotes the Newtonian friction at the given Reynolds number and $\Delta B = \alpha \ln(v^*/v_0^*)$ as indicated in Eq. (2). Thus the polymer additive causes an increase in $f^{-1/2}$ equal to 71% of the upward shift ΔB of the law of the wall curve. This is illustrated in Fig. 2.

Secondly, if Eq. (3) is rearranged to solve for the Reynolds number, the result is that the friction factor in the presence of the additive is equal to the Newtonian friction factor evaluated at an effective Reynolds number R_N given by

$$R_N = R(v^*/v_0^*)^{k\alpha} \quad (5)$$

Thus the polymer has the effect of increasing the Newtonian Reynolds number and thereby, of course, decreasing the friction factor. This effective increase in R is also sketched in Fig. 2.

Boundary-Layer Analysis

To derive a formula for boundary-layer friction on a flat plate that is similar to the pipe expression of Eq. (3), it is only necessary to assume, as is done in Newtonian theory, that the law of the wall as modified by Meyer is still applicable:

$$u^+ = (1/k) \ln(y^+) + 5.5 + \alpha \ln(v^*/v_0^*) \quad (6)$$

where α and v_0^* are given, say, from Fig. 1. Unlike the pipe, the boundary-layer shear-stress distribution is not known a priori, but may nevertheless be given exactly by an extension of the elegant method proposed by Brand and Persen.[10] From Eq. (6), the velocity component $u(x, y)$ for two-dimensional flow is known, since, by definition,

$$u^+(y^+) = u(x,y)/v^*(x) \quad (7)$$

Since $u(x, y)$ is specified, the velocity $v(x, y)$ may immediately be calculated from the incompressible continuity equa-

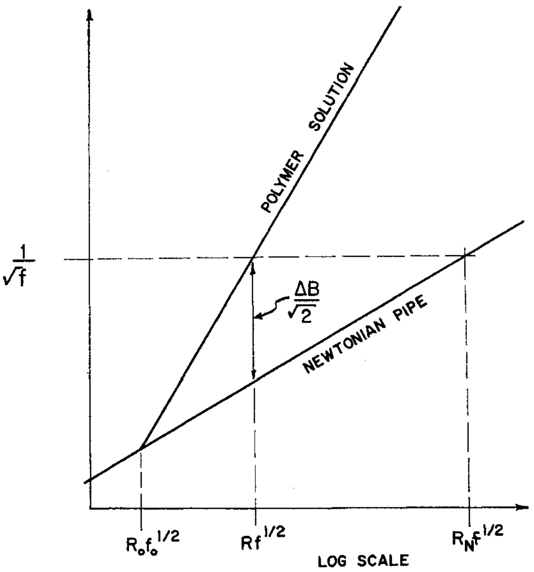

Fig. 2 Illustration and interpretation of polymer pipe friction reduction.

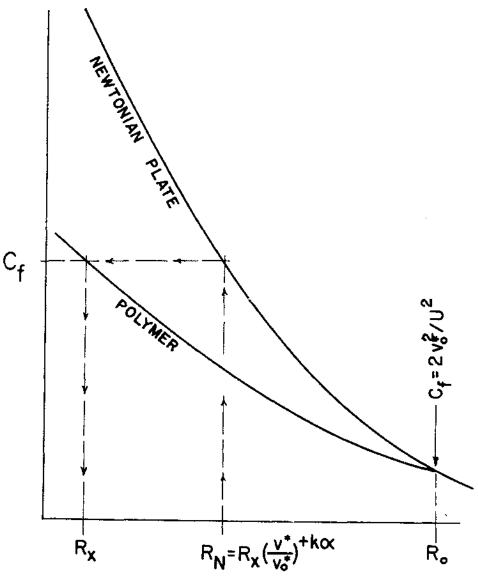

Fig. 3 Illustration of the effect of a polymer on flat-plate skin friction.

tion for two-dimensional mean flow, or

$$v = -(\nu/v^*)(dv^*/dx)y^+(u^+ + \alpha) \quad (8)$$

The two-dimensional boundary-layer equation may be written as

$$\rho\ [u(\partial u/\partial x) + v(\partial u/\partial y)] = -(dp/dx) + (\partial\tau/\partial y) \quad (9)$$

The convective acceleration is calculated by substituting u and v from Eqs. (6–8),

$$u(\partial u/\partial x) + v(\partial u/\partial y) = v^*(dv^*/dx)\ (u^{+2} + u^+ - \alpha/k) \quad (10)$$

We introduce Eq. (10) into Eq. (9) and set $dp/dx = 0$ for the flat plate, since Eq. (10) is not accurate for any but the mildest pressure gradient.[11] The momentum equation, Eq. (9), may then be integrated from the wall to the outer edge of the boundary layer with the following result:

$$\tau_w = -\rho\nu\frac{dv^*}{dx}\left[\int_0^{\delta^+}(u^{+2} + \alpha u^+)dy^+ - \frac{\alpha}{k}\delta^+\right] \quad (11)$$

where δ^+ is the value of y^+ at the outer edge of the boundary layer. Letting U be the freestream velocity and introducing the notation,

$$\xi = \frac{U}{v^*} \qquad G(\xi) = \int_0^{\delta^+}(u^{+2} + \alpha u^+)dy^+ \quad (12)$$

Eq. (11) then becomes

$$(U/\nu)dx = d\xi[G - (\alpha/k)\delta^+] \quad (13)$$

which may be integrated immediately to yield a relationship between Reynolds number R_x and the local plate friction factor $C_f = 2\tau_w/\rho\ U^2$. The required integrals of G and δ^+ in Eq. (13) are found from the assumed relationship $u^+(y^+)$ in Eq. (6). Then

$$R_x = \frac{Ux}{\nu} = \frac{0.1108}{k^3}e^{k\xi}\left(\frac{v^*}{v_0^*}\right)^{-k\alpha} \times$$
$$[k^2\xi^2 - 4k\xi + 6 + k\alpha(k\xi - 3)] \quad (14)$$

which is the central result of the present paper. The quantity ξ is equivalent to the friction coefficient C_f, since, from Eq. (12),

$$\xi = U/v^* = (2/C_f)^{1/2} \quad (15)$$

When $\alpha = 0$, Eq. (14) reduces to the Newtonian skin-friction relation for a flat plate, which was first given in this form by Kestin and Persen.[12]

The term $k\alpha(k\xi - 3)$ within the square bracket of Eq. (14) is not large, since α is a maximum of about ten, so that the term contributes a maximum of 35% of the bracket value at low Reynolds numbers and 20% at high Reynolds numbers. More important, the term is suspect physically, since it arises from the differentiation of Meyer's correlation, Eq. (6), which has a discontinuity at the threshold point. It is suggested here that the term, being both small and spurious, should be neglected.

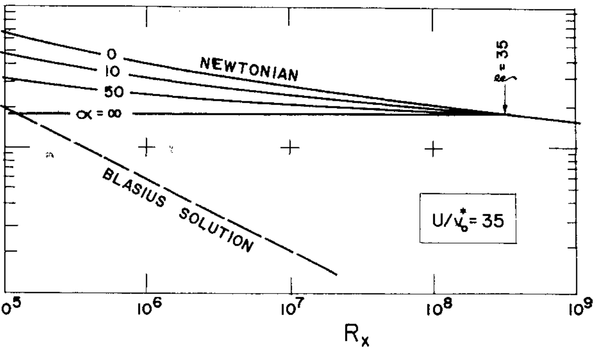

Fig. 4 Effect of the polymer constant α on flat-plate friction for a speed ratio $U/v_0^* = 35$ [Eq. (14)].

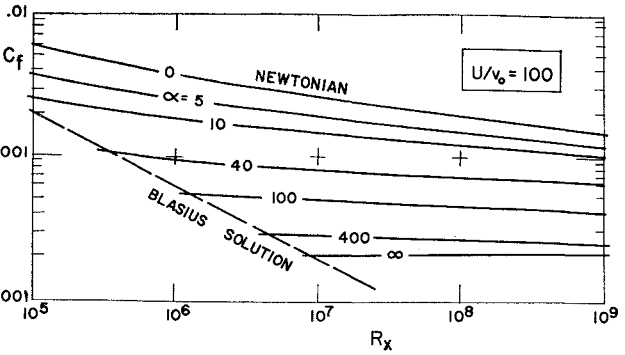

Fig. 5 Effect of the polymer constant α on flat-plate friction for a speed ratio $U/v_0^* = 100$ [Eq. (14)].

Equation (14) then is subject to the following interpretation: the local skin friction C_f on a flat plate with polymer additive is equal to the Newtonian skin friction evaluated at an effective Reynolds number R_N given by

$$R_N \doteq R_x(v^*/v_0^*)^{k\alpha} \quad (16)$$

which is identical to the pipe flow relation, Eq. (5). However, in applying this "effective Reynolds number" concept, the plate behaves rather in an opposite manner to the pipe, because of the different manner in which v^* varies with Reynolds number. In the pipe, v^* increases with Reynolds number, so that the polymer causes a skin-friction reduction at *large* Reynolds numbers, as shown in Fig. 1. On a flat plate, however, v^* decreases with R_x, since U remains constant; hence, the polymer reduces plate friction only at *small* Reynolds numbers. This effect for the plate is sketched in Fig. 3.

Here, as with the pipe, the polymer and Newtonian curves intersect and merge at a Reynolds number R_0 where the friction velocity v^* equals the threshold value v_0^*. For the plate, we should note that

$$(v^*/v_0^*)_{\text{plate}} = (U/v_0^*)(C_f/2)^{1/2} \quad (17)$$

According to the present theory, then, polymer friction reduction for a plate is confined to its leading edge, where R_x is less than R_0. This means that the integrated drag of a long plate, where $R_L \gg R_0$, is not reduced very much by a polymer additive, since most of the friction is below the threshold value v_0^* at which the polymer becomes "excited."

To obtain numerical values of local plate skin friction C_f with a polymer additive, one may either use Eq. (14), which needs iteration if R_x is assumed known, or the effective Reynolds number R_N from Eq. (16) together with some simple Newtonian correlation, such as that of Schlichting[13]:

$$C_f(\text{Newtonian}) = [2\log(R_N) - 0.65]^{-2.3} \quad (18)$$

Such numerical results, taken from Eq. (14), are illustrated in Figs. 4–6. Two convenient parameters are the polymer constant α from Eq. (6) and the speed ratio (U/v_0^*). Figure 4 is for a small speed ratio of 35.0 and shows that the effect of the polymer is then rather slight, even for large α, since the additive remains relatively unexcited. The drag reduction at this low speed obviously is not large. Note that the minimum value of C_f achieved by the polymer (when $\alpha = \infty$) is equal to 2.0 divided by the square of the speed ratio, i.e., the value of C_f at $R_x = R_0 = 3 \times 10^8$ for this case.

Figure 5 shows the opposite case with a large value of (U/v_0^*). Here the skin-friction reduction is substantial in all ranges, since the intersection Reynolds number is extremely large, $R_0 \doteq 10^{20}$. The integrated drag reduction is also substantial, since R_0 is much larger than any practical-body Reynolds number R_L.

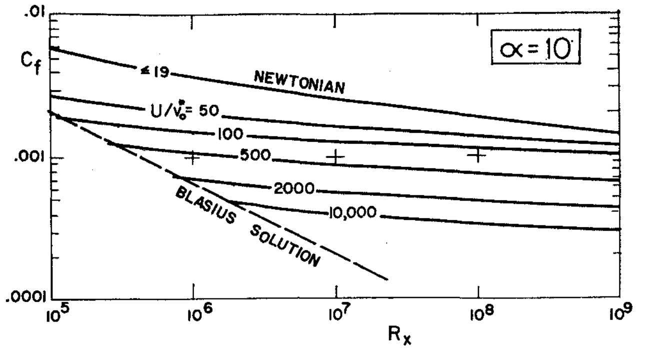

Fig. 6 Effect of speed ratio (U/v_0^*) on flat-plate local skin friction for $\alpha = 10$ [Eq.(14)].

Figure 6 shows the effect of speed ratio for $\alpha = 10$, which is, from Fig. 1, near maximum for present polymers. The indicated friction reductions are huge if the speed ratio is large, but it is debatable if Meyer's correlation, Eq. (6), persists for such large shear. Tentative results by McNally[14] for Polyox WSR-301 show a leveling-off of the polymer effect at large shear rates, indicating that Eq. (2) cannot be indefinitely extrapolated.

Integrated Flat-Plate Drag

The drag coefficient of one side of the plate is given by the familiar formula

$$C_D = \frac{2\,(\text{drag})}{\rho U^2 L} = \int_0^1 C_f d\,\frac{x}{L} \qquad (19)$$

Since Eq. (14) is implicit in C_f, it is very inconvenient to use in evaluating Eq. (19), although numerical integration is of course straightforward. An approximate expression for C_D is found from the effective Reynolds number, Eq. (16), and a power-law expression for the Newtonian skin friction†:

$$C_f(\text{Newtonian}) \doteq 0.026/R_N^{1/7} \qquad (20)$$

Equation (20) may be combined with the effective Reynolds number concept from Eq. (16) to yield an approximate expression for local skin friction with the polymer additive:

$$C_f \doteq A/(R_x)^{2/(14+k\alpha)}$$
$$A = [0.0260(2v_0^{*2}/U^2)^{k\alpha/14}]^{1/(1+k\alpha/14)} \qquad (21)$$

Equation (21) holds when R_x is less than R_0, while, otherwise, Eq. (20) holds. These two relations may be substituted into Eq. (19) to yield an approximate drag coefficient,

$$C_D \doteq A\,\frac{(14+k\alpha)}{(12+k\alpha)}\,\frac{(R_0/R_L)^{(12+k\alpha)/(14+k\alpha)}}{R_L^{2/(14+k\alpha)}} + \frac{0.0303}{R_L^{1/7}}\,[1-(R_0/R_L)^{6/7}] \qquad (22)$$

If R_0 is greater than R_L, the ratio (R_0/R_L) is taken equal to unity. Equation (22) is fairly accurate and is simple to use to estimate the percentage drag reduction of a given additive and body length when α and v_0^* are known. Figure 7 shows the drag reduction for $\alpha = 10$ as calculated by Eq. (22). The polymer effectiveness is markedly decreased at large plate-length Reynolds numbers, in accordance with the confinement of polymer excitation to the leading edge. The decrease of drag reduction with R_L is especially sharp at low-speed ratios, where R_0 is of the order of R_L. The dotted lines in Fig. 7 indicate the possibility of a large amount of laminar flow, where Eq. (22) would not apply. We have seen in Figs. 4–6 that the theoretical polymer curves appear to cross at an angle to the familiar Blasius laminar curve ($C_f = 0.664/R_x^{1/2}$). This is not unrealistic, since Davis[2] has shown that the laminar plate skin friction is also reduced by non-Newtonian effects. However, it is probable that the actual curves turn upwards at low Reynolds numbers, since Meyer's correlation, Eq. (2), is not accurate at the arbitrarily large leading-edge shear rates.

Effect of Pressure Gradient

The preceding analysis led to Eq. (14) through an exact integration of the turbulent boundary-layer equation, Eq. (9), which depends only upon the correctness of Meyer's velocity correlation, Eq. (6). It is accurate only for a flat plate, since Eq. (6) almost surely does not hold in a region of significant pressure gradient. For pressure gradient flows, Clauser[15] has shown that the velocity profiles are dependent upon the dimensionless equilibrium parameter

$$\beta = (\delta^*/\tau_w)(dp/dx) \qquad (23)$$

where δ^* is the displacement thickness. Deviations from Eq. (6) are important if β is of the order of unity. For pipe flow, $\beta \doteq -0.2$, nearly independent of Reynolds number, so that a pipe is essentially equivalent to flat-plate flow.

The present analysis may be extended to pressure gradients by incorporating an additional velocity term into Eq. (6).

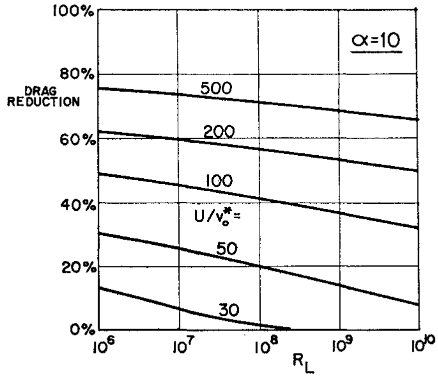

Fig. 7 Theoretical total drag reduction of a flat plate for a polymer constant $\alpha = 10$ [Eq. (22)].

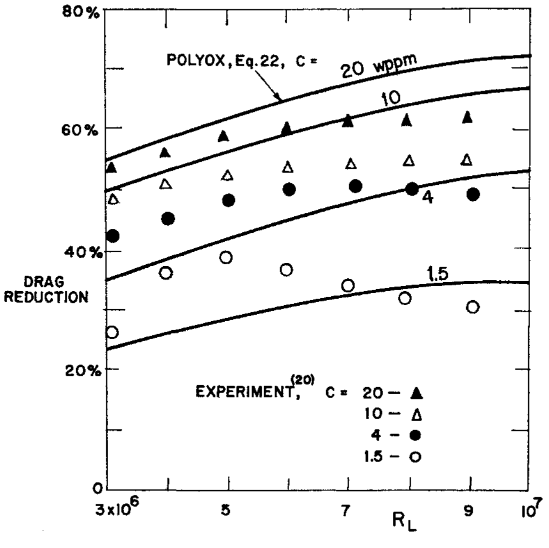

Fig. 8 Comparison of the present theory with the thin-plate experiment of Levy.[20]

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† This formula is much more accurate than Eq. (21.12) of Ref. 13.

This has been done for the Newtonian case by White,[11] who incorporated an empirical term based upon the ideas of Mellor.[16] Eq. (6) as modified according to White's suggestion is

$$u^+ = u^+]_{(dp/dx)=0} + \frac{0.6\nu}{\tau_w v^*}\frac{dp}{dx}y^+ \qquad (24)$$

When Eq. (24) was used for the Newtonian case ($\alpha = 0$) in Ref. 11, the result was a first-order ordinary differential equation for the skin-friction parameter $\xi(x)$, similar to Eq. (13). Agreement with experiment was good, particularly for separating flows. An analysis in the present case would be entirely similar except for the added polymer terms, but the details will not be given until the flat-plate theory can be compared thoroughly with experiment. It should be mentioned that, when pressure gradients occur, the effective Reynolds number concept of Eq. (16) is not directly applicable, since the solution is dependent upon the integrated "history" of the variable freestream velocity $U(x)$.

Comparison with Experiment

Experimental data have been reported by Levy[20] and Kowalski[24] for a thin plate moving through a dilute homogeneous solution of Polyox WSR-301. Although polyox degrades rapidly with both time and use, these studies afford a reasonable basis for comparison with the present theory. Figure 8 shows such a comparison of Eq. (22) with the data of Levy[20] for polyox concentrations $C = 1.5, 4, 10$, and 20 wppm. The values of α for Eq. (22) were calculated from the Fig. 1 curve-fit $\alpha = 2.3C^{1/2}$. The agreement is fairly good, with discrepancies probably due to 1) the fact that Eq. (22) is based on a crude power-law approximation, and 2) the data suffer from polymer degradation and from high shear stresses which are beyond the limit of Meyer's logarithmic correlation, Eq. (6).

The accuracy of the theory can be improved if we discard Meyer's parameter α and work instead with the parameter ΔB which denotes the upward shift of the law of the wall. Pipe flow could be correlated by Eq. (4), whereas Eq. (16) would be written in the equivalent form

$$R_N \doteq R_x e^{-k\Delta B} \qquad (25)$$

The simple power-law theory, Eq. (21), becomes

$$C_f = (0.026/R_x^{1/7})\, e^{-(k/7)\Delta B} \qquad (26)$$

By comparing these expressions with experiment, values of ΔB can be calculated and correlated with polymer concentration and wall shear stress, including degradation effects to some extent. The validity of this alternate ΔB correlation is very promising, as pointed out in a recent lecture by A. J. Fabula.[25]

Conclusions

Since Meyer's law of the wall correlation, either in its α form or its ΔB form, is apparently quite reasonable for known polymer additives, the present exact solution for flat-plate flow, Eq. (14), is plausible and justifiable as a theory. However, the analysis is subject to some criticism when compared to the practical situation in boundary layers, as follows:

1) The analysis assumes the entire fluid to be a uniform polymer solution. In practice, the polymer might be injected near the leading edge, with resultant nonuniform downstream concentration. At high velocities and/or concentrations, diffusion is poor and the polymer effectiveness less than predicted. This was the case in the experiment of Love,[17] who reported decreased drag reduction at high velocities and high injection concentrations, probably due to poor diffusive mixing and excessively thick injection solution.

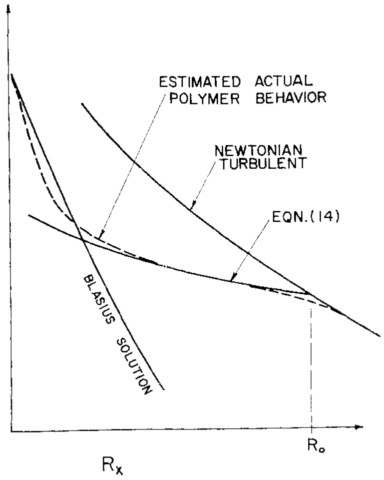

Fig. 9 Sketch of the theoretical and estimated actual behavior of a polymer solution in flat-plate flow.

2) The theory is generalized from pipe flow experiments where the wall shear stress is constant. To apply this relation to boundary-layer flow, the analysis assumes that the polymer can accommodate itself immediately and reversibly to the decreasing wall shear as it moves downstream. This is probably inaccurate both at the leading edge, where the shear is very high, and far downstream at the threshold, where the polymer effectiveness probably persists past the point where $v^* = v_0^*$. Figure 9 is an educated guess as to the actual local skin friction, compared to the prediction of Eq. (14).

3) Equation (14) is valid only for a smooth wall, whereas actual plates may be hydraulically rough. However, it is possible that the effective Reynolds number concept, Eq. (16) or (25), may be valid if the skin friction for R_N is referred to the rough Newtonian wall case. McNally[14] has reported that a polymer is still very effective in a rough pipe, whereas Love[17] shows only very slight polymer drag reduction in a rough plate experiment.

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